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Derivation and quantisation of Solov'ev's constant for the diamagnetic Kepler motion

M Kuwata, A Harada and H Hasegawa

Department of Physics, Kyoto University, Kyoto 606, Japan

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Abstract. We report that the hyperbolic form of the Runge-Lenz vector $\boldsymbol{A}$, i.e. $\Lambda(\boldsymbol{A}) = 4A^2 - 5A_z^2$, which was shown to be an approximate constant of the diamagnetic Kepler motion by Solov'ev, can be deduced as the lowest non-trivial Birkhoff-Gustavson normal form for a resonant set of generally four oscillators subject to a constraint. Its special case $p_\phi(=m) = 0$ (i.e. the case of vanishing magnetic quantum number) is shown to agree with the result of Robnik and Schrüfer derived from two oscillators. A systematic scheme of the semiclassical quantisation (the torus quantisation) is discussed on the explicit construction of the genus-one topology, by means of which all the possible, equivalent quantisation formulae are deduced.

1. Introduction

A great deal of interest has been shown recently in studies of quantum spectra of the hydrogen atom subject to a magnetic field in connection with the correspondence problem between classical dynamics and quantum mechanics, in particular, with the problem of *chaos* (see a number of works reported in the book edited by Taylor (1988)). One of the motives for this interest was the paper by Solov'ev (1981) who established for the first time the explicit expression of the notion *approximate constant of motion* which had been anticipated to exist in the Hamiltonian function pertinent to the problem (Zimmerman *et al* 1980, Clark and Taylor 1980, Clark 1981, Robnik 1981; see also a recent review by Hasegawa *et al* 1989).

An approximate constant of the motion associated with a series-expanded Hamiltonian function, $H = \Sigma_{i=2}^{\infty} H^{(i)}(p, q)$ canonically transformed from a given function, is defined by the quantity $\Lambda^{(i)}(p, q)$ whose Poisson bracket with the jth truncated part $(i \leqslant j)$ $H^{(j)} = \Sigma_{i=2}^{j} H^{(i)}(p, q)$ of H vanishes, where the lowest term $H^{(j=2)}$ (the unperturbed H) is assumed to represent a set of harmonic oscillators. Such was initiated by Birkhoff (1927) for non-resonant oscillators (i.e. n harmonic oscillators whose frequencies $\omega_1, \omega_2, \ldots, \omega_n$ satisfy no linear relations $\Sigma\, a_\nu \omega_\nu = 0$ with integer coefficients a_ν; note, in this case $\Lambda^{(j)} = H^{(j)}$ only), and later generalised by Gustavson (1966) for resonant oscillators, say rth fold resonance (i.e. r linear independent relations $\Sigma\, a_{\sigma\nu}\omega_\nu = 0$ exist with integer coefficients $a_{\sigma\nu}$). Gustavson's work was concerned with a concrete example of the Hénon-Heiles pair of oscillators for which $n = 2$ and the equal frequencies, the so-called 1:1 resonance, must yield generally first-fold $(r = 1)$ resonance for every truncated Hamiltonian $H^{(j)}$, $j > 2$. It ensures one approximate constant in addition to the unperturbed Hamiltonian of the form $\Sigma_{\nu=1}^{2} \frac{1}{2}(p_\nu^2 + \omega^2 q_\nu^2)$. An important significance of Gustavson's theorem is that such approximate constants that exist yield the analytic curves to simulate the invariant tori of a non-separable dynamics in the context of Kolmogorov, Arnold and Moser (the so-called KAM theorem; see Lichtenberg and Lieberman 1983).

The discovery of the approximate constant for the diamagnetic Kepler Hamiltonian in terms of the Runge-Lenz vector $\boldsymbol{A}$ of the form $\Lambda(\boldsymbol{A}) = 4A^2 - 5A_z^2$ by Solov'ev (1982), which was not made in the above context, should therefore be subject to theoretical investigation as to whether it can be deduced by the general procedure of Gustavson. The purpose of this paper is to discuss an affirmative answer to this question. We note here that the previous reports on the same problem have been either by numerical demonstrations (Reinhardt and Farrelly 1982, Hasegawa *et al* 1984, Saini and Farrelly 1987) or, a precise derivation of the normal form but without relating it to $\Lambda(\boldsymbol{A})$ (Robnik and Schrüfer 1985). Our result in section 2 shows that the diamagnetic Kepler Hamiltonian can be converted into that of a system of four oscillators with equal frequencies (i.e. 1:1:1:1 resonance) subject to a constraint, thus reducing the degree of freedom by one, whose lowest non-trivial normal form agrees precisely with that predicted by Solov'ev and the approximate constant with Solov'ev's form $\Lambda(\boldsymbol{A})$.

In section 3 we treat the special situation $p_\phi = 0$ where p_ϕ is another (exact) constant, i.e. the angular momentum component along the magnetic field. We show that the normal form then reduces to the result of Robnik and Schrüfer (1985): in this situation, the initial four-oscillator Hamiltonian reduces to that of the two oscillators in the parabolic coordinate system adopted by the above authors, which yields a systematic analysis of the choice of the action and angle variables. In the lowest-order non-trivial normal form, every action integration is shown to be expressed in an elliptic integral, implying that the underlying KAM torus is typically of the topology with genus one (i.e. the doughnut structure). Moreover, the action integral to this order can be represented as the Cauchy integral on a complex Riemann surface which allows several equivalent but different forms of the representation. We show that the quantum number entering the quantisation formula must be a half-odd integer. Also, we clarify some puzzling features of the formula that Solov'ev proposed.

2. The normal-form Hamiltonian with four oscillators

We follow Kustaanheimo and Stiefel (1965) by introducing a mapping of a four-dimensional space R^4 with local coordinates (u_1, u_2, u_3, u_4) onto the physical space R^3 with coordinates (x_1, x_2, x_3) by means of the transformation T (hereafter called the KS transformation):

$$\boldsymbol{x} = T\boldsymbol{u} \qquad T = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{x}$ are two column vectors in R^4

$$\boldsymbol{u} = {}^{\mathrm{t}}(u_1, u_2, u_3, u_4) \qquad \boldsymbol{x} = {}^{\mathrm{t}}(x_1, x_2, x_3, 0) \tag{2}$$

and T satisfies orthogonalities

$$T\,{}^{\mathrm{t}}T = {}^{\mathrm{t}}TT = |\boldsymbol{u}|^2 1 \tag{3}$$

(^tT denotes the transpose of T, and $|\boldsymbol{u}|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2$). The fourth component of $\boldsymbol{x}$ as the image of the map of $\boldsymbol{u}$ identically vanishes, and hence the nonlinear relations are introduced, i.e.

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 \qquad x_2 = 2(u_1u_2 - u_3u_4) \qquad x_3 = 2(u_1u_3 + u_2u_4)$$

defines a map $R^4 \to R^3$ which is *onto* but not *one-to-one*. Accordingly, the kernel of the map forms a one-parameter family, and Kustaanheimo and Stiefel (1965) showed that this family is identified with a subgroup of rotations of the form

$$\boldsymbol{u} \to \boldsymbol{u}' = R_\phi \boldsymbol{u} \qquad R_\phi = \begin{pmatrix} \cos\phi & 0 & 0 & -\sin\phi \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ \sin\phi & 0 & 0 & \cos\phi \end{pmatrix} \tag{4}$$

or, in other words, the inverse map $R^3 \to R^4$ can be defined up to the equivalence class of $\boldsymbol{u}$ vectors in R^4 with $\boldsymbol{u}' \equiv \boldsymbol{u}$ by the above type of rotations.

For classical dynamics of a particle in R^3, when described by the $\boldsymbol{u}$ coordinates with four degrees of freedom, the above equivalence class can be characterised by

$$-u_4p_1 + u_3p_2 - u_2p_3 + u_1p_4 = 0 \tag{5a}$$

where

$$\boldsymbol{p}_u = {}^t(p_1p_2p_3p_4) \tag{5b}$$

is a canonical momentum vector conjugate to $\boldsymbol{u}$. It says that the angular momentum L_ϕ associated with the rotation (4) vanishes, or, in terms of the ordinary angular momentum components L_{ij} in R^4, that

$$L_{14} = L_{23} \qquad u_1p_4 - u_4p_1 = u_2p_3 - u_3p_2 \tag{6}$$

which is shown to ensure the identity

$$\sum_{i=1}^{3} p_{x_i}\,\mathrm{d}x_i = \sum_{i=1}^{4} p_i\,\mathrm{d}u_i \tag{7}$$

that is to say, the transformation between the phase spaces $R^3 \times R^3$ and $R^4 \times R^4$ under the constraint (5) or (6) is canonical. For more detailed properties of the constraint, see Boiteux (1973) and Cornish (1984a, b). The canonicity (7) together with the KS transformation (1) enables us to write

$$\boldsymbol{p}_x = \frac{1}{2r} T\boldsymbol{p}_u \tag{8a}$$

where

$$\boldsymbol{p}_x = {}^t(p_{x_1}p_{x_2}p_{x_3}0). \tag{8b}$$

Let us now write the diamagnetic Kepler Hamiltonian as

$$H = \frac{1}{2}\boldsymbol{p}_x^2 + \frac{\gamma}{2}p_\phi + \frac{\gamma^2}{8}(\boldsymbol{x}^2 - (\boldsymbol{x}\cdot\hat{\boldsymbol{\gamma}})^2) - \frac{1}{r} \tag{9}$$

where $\hat{\gamma}$ = unit vector along the magnetic field $\boldsymbol{B}$, γ = strength of the magnetic field in atomic units ($B_0 = 2.35 \times 10^9$ Gauss), p_ϕ = component of the angular momentum $\boldsymbol{L}$ along $\boldsymbol{B}$ (constant of the motion).

An application of the KS transformation (1) and its momentum version (8) to this Hamiltonian is shown, when x_1 is chosen as the coordinate along the magnetic field, to yield an unharmonic oscillator Hamiltonian of the form

$$H = \frac{1}{8r} \boldsymbol{p}_u^2 - \frac{1}{|\boldsymbol{u}|^2} + \frac{\gamma}{2} p_\phi + \frac{\gamma^2}{2} (u_1^2 + u_4^2)(u_2^2 + u_3^2). \tag{10}$$

If, further, the time variable t of this dynamics is changed to s through

$$\left(\frac{dt}{ds}\right)^2 = 4r = 4|\boldsymbol{u}|^2 \tag{11}$$

the Hamiltonian (10) is multiplied by the above factor to eliminate the Coulomb singularity, which is called *regularisation* (the so-called Levi-Civita regularisation, see Reinhardt and Farrelly 1982). The stationary condition of the classical dynamics on an energy surface $H = E$ then becomes

$$\mathscr{H} \equiv \tfrac{1}{2}(\boldsymbol{p}_u^2 + \omega^2 \boldsymbol{u}^2) + 2\gamma^2 |\boldsymbol{u}|^2 (u_1^2 + u_4^2)(u_2^2 + u_3^2) = 4 \tag{12a}$$

with

$$\omega^2 = 4\gamma p_\phi - 8E \quad (>0 \text{ for a bounded Kepler motion}) \tag{12b}$$

the expression $\mathscr{H}$ being regarded as a Hamiltonian of u oscillators. It comprises the harmonic unperturbed part plus the sextet unharmonicity due to the diamagnetism, thus allowing a direct application of the Birkhoff–Gustavson procedure. It can be seen from the unperturbed part $\mathscr{H}^{(2)}$ that the problem of normalisation belongs to Gustavson's third fold (1:1:1:1) resonance, and hence three extra constants of the motion may exist (one is reduced by virtue of the constraint (5) and another is identified with p_ϕ $(=L_{14} = L_{23}$ in (6)). Hereafter, the linear Zeeman term is omitted by absorbing it into the energy E.

We need to determine the part $\mathscr{H}_{\rm NF}$ from $\mathscr{H}$ which meets the normal form condition:

$$D\mathscr{H}_{\rm NF} \equiv \{\mathscr{H}_{\rm NF}, \mathscr{H}^{(2)}\} = 0 \tag{13}$$

where the symbol $\{,\}$ denotes the Poisson bracket and the operator D

$$D\cdot = \{\cdot, \mathscr{H}^{(2)}\} = \sum_{j=1}^{4} \left(p_j \frac{\partial}{\partial u_j} - \omega^2 u_j \frac{\partial}{\partial p_j} \right). \tag{14}$$

It is useful to introduce the complex variables

$$z_j = \frac{1}{\sqrt{2\omega}} (\omega u_j + \mathrm{i} p_j) \qquad z_j^* = \text{cc of } z_j \quad j = 1, \ldots, 4 \tag{15}$$

so that the operator D can now be expressed as

$$D = -\mathrm{i}\omega \sum_{j=1}^{4} \left(z_j \frac{\partial}{\partial z_j} - z_j^* \frac{\partial}{\partial z_j^*} \right). \tag{16}$$

The kernel of $D (= \{f \mid Df = 0\})$ can be determined by monomials $z^m z^{*n} \equiv \prod_j z_j^{m_j} z_j^{*n_j}$ such that

$$D z^m z^{*n} = -\mathrm{i}\omega \sum_j (m_j - n_j) z^m z^{*n} = 0$$

that is, the monomial $z^m z^{*n}$ belongs to ker D, if and only if

$$\sum_j (m_j - n_j) = 0. \tag{17}$$

The formulation is based on Robnik (1984).

The sextet unharmonic part of the Hamiltonian in (12*a*), i.e.

$$\mathscr{H}^{(6)} = 2\gamma^2 |\boldsymbol{u}|^2 (u_1^2 + u_4^2)(u_2^2 + u_3^2)$$

is then shown to have its normal form

$$\mathscr{H}_{\rm NF}^{(6)} = \frac{\gamma^2}{\omega^3}\Bigg[(|z_2|^2 + |z_3|^2)|z_1^2 + z_4^2|^2 + (|z_1|^2 + |z_2|^2)|z_2^2 + z_3^2|^2 + \left(\sum_{i=1}^4 |z_i|^2\right) \times \{(|z_1|^2 + |z_4|^2)(|z_2|^2 + |z_3|^2) + \mathrm{Re}(z_1^2 + z_4^2)(z_2^{*2} + z_3^{*2})\}\Bigg]$$

which reduces to

$$\mathscr{H}_{\rm NF}^{(6)} = \frac{\gamma^2}{\omega^3}\left(\sum_{j=1}^4 |z_j|^2\right)\{5(|z_1|^2 + |z_4|^2)(|z_2|^2 + |z_3|^2) + L_z^2 - 4\boldsymbol{L}^2\} \tag{18}$$

by using the two relations for the angular momentum $\boldsymbol{L}$ in R^3:

$$2\boldsymbol{L}^2 = (|z_1|^2 + |z_4|^2)(|z_2|^2 + |z_3|^2) - \mathrm{Re}(z_1^2 + z_4^2)(z_2^{*2} + z_3^{*2}) + L_z^2 \tag{19a}$$

$$L_z^2 (= p_\phi^2) = (|z_1|^2 + |z_4|^2)^2 - |z_1^2 + z_4^2|^2 = (|z_2|^2 + |z_3|^2)^2 - |z_2^2 + z_3^2|^2 \qquad \hat{z} = \hat{\gamma} /\!/ \boldsymbol{B} \tag{19b}$$

(the latter two identical expressions stem from the constraint (5*a*) or (6)). Or, in terms of the canonical variables in R^4,

$$\mathscr{H}_{\rm NF}^{(6)} = \frac{\gamma^2}{\omega^6}\frac{1}{2}(\boldsymbol{p}_u^2 + \omega^2 \boldsymbol{u}^2) \times \{5 \cdot \tfrac{1}{2}[p_1^2 + p_4^2 + \omega^2(u_1^2 + u_4^2)] \cdot \tfrac{1}{2}[p_2^2 + p_3^2 + \omega^2(u_2^2 + u_3^2)] + \omega^2(L_z^2 - 4\boldsymbol{L}^2)\}. \tag{20}$$

Expression (20) is now connected with the Runge–Lenz vector $\boldsymbol{A}$ of the Kepler motion defined by some vectors in R^3. The key formulae for this connection are:

$$\boldsymbol{A}^2 = 1 - \frac{\omega^2}{4}\boldsymbol{L}^2 \qquad \boldsymbol{A} \equiv \boldsymbol{P}_x \times \boldsymbol{L} - \frac{\boldsymbol{x}}{r^2} \tag{21}$$

and

$$1 - A_z = \tfrac{1}{4}(p_1^2 + p_4^2) + \tfrac{1}{4}\omega^2(u_1^2 + u_4^2)$$
$$1 + A_z = \tfrac{1}{4}(p_2^2 + p_3^2) + \tfrac{1}{4}\omega^2(u_2^2 + u_3^2). \tag{22}$$

The latter two identities can be deduced from the KS transformation and its momentum versions (1) and (8*a*) applied to A_z, if it is recalled that the magnetic axis z is identical to the first component x_1 in R^3. Thus, by inserting these two identities in (22) and the relation (21), we get

$$\mathscr{H}_{\rm NF} = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{u}^2)\left[1 + \frac{4\gamma^2}{\omega^6}\left\{1 + \Lambda(\boldsymbol{A}) + \frac{\omega^2}{4}p_\phi^2\right\}\right] + \mathrm{O}(\gamma^4) \tag{23a}$$

where

$$\Lambda(\boldsymbol{A}) = 4\boldsymbol{A}^2 - 5A_z^2 \tag{23b}$$

(a Runge–Lenz hyperboloid). Up to this order of truncating the normal form of $\mathscr{H}$, therefore, the Runge–Lenz hyperboloid $\Lambda(\boldsymbol{A})$ given by (23b) which is involutive with $\mathscr{H}^{(2)} = \frac{1}{2}(\boldsymbol{p}^2 + \omega^2\boldsymbol{u}^2)$, as well as with p_ϕ, represents an approximate constant of the Kepler motion. Also, in terms of the starting Kepler Hamiltonian (9) (following the argument of Robnik and Schrüfer 1985),

$$H_{\rm NF} = -\tfrac{1}{2}(\mathscr{H}_{\rm NF})^{-2} = -\tfrac{1}{2}(\mathscr{H}_{\rm NF}^{(2)})^{-2}\left[1 + \frac{4\gamma^2}{\omega^6}\left\{1 + \Lambda(\boldsymbol{A}) + \frac{\omega^2}{4}p_\phi^2\right\}\right]^{-2}$$

$$= -\frac{1}{2N^2}\left[1 - \frac{2\gamma^2}{16}N^6\left\{1 + \Lambda(\boldsymbol{A}) + \left(\frac{p_\phi}{N}\right)^2\right\}\right] \qquad N \equiv (-2E)^{-1/2}.$$

It implies that the diamagnetic correction to the Rydberg energy is given by

$$H_d^{(2)} = \frac{\gamma^2}{16}N^4\left\{1 + \Lambda(\boldsymbol{A}) + \left(\frac{p_\phi}{N}\right)^2\right\} \tag{24}$$

which lifts the degenerate Rydberg multiplet with energy $-1/2N^2$. This completes the derivation of Solov'ev's result by means of the Birkhoff–Gustavson normalisation procedure.

3. Case for $p_\phi = 0$ and the semiclassical quantisation

As we have mentioned already, the angular momentum component $L_z(=p_\phi)$ is identical to L_{14} $(=L_{23})$, whence

$$L_z = \mathrm{i}(z_1 z_4^* - z_1^* z_4) = \mathrm{i}(z_2 z_3^* - z_2^* z_3) = 0$$

implies that in the polar representation of the z variables

$$z_j = \sqrt{I_j}\,\mathrm{e}^{\mathrm{i}\phi_j} \qquad j = 1, 2, 3, 4 \tag{25}$$

the following identities hold:

$$\phi_1 = \phi_4 \qquad \text{and} \qquad \phi_2 = \phi_3. \tag{26}$$

In this situation, the two two-dimensional oscillators composed of (u_1, u_4) and (u_2, u_3) become one-dimensional ones: there exist two cartesian coordinates (u, v) and the respective conjugate momenta (p_u, p_v) such that

$$u^2 = u_1^2 + u_4^2,\ p_u^2 = p_1^2 + p_4^2 \qquad \text{and} \qquad v^2 = u_2^2 + u_3^2,\ p_v^2 = p_2^2 + p_3^2$$

with which (12a) becomes

$$\mathscr{H} = \tfrac{1}{2}(p_u^2 + \omega^2 u^2) + \tfrac{1}{2}(p_v^2 + \omega^2 v^2) + 2\gamma^2(u^2 + v^2)u^2 v^2. \tag{12a'}$$

Also, $1 \pm A_z$ associated with the z component of the Runge–Lenz vector expressed in (22) becomes

$$1 - A_z = \tfrac{1}{4}(p_u^2 + \omega^2 u^2) \qquad 1 + A_z = \tfrac{1}{4}(p_v^2 + \omega^2 v^2). \tag{22'}$$

Expression (12a') is the diamagnetic Kepler Hamiltonian converted from (10) for $p_\phi = 0$ in terms of the parabolic coordinate system, and (apart from a scaling factor) is identical to the one adopted by Robnik and Schrüfer (1985). Thus, one expects that our result (24) with $p_\phi = 0$ should be contained in their BG normalisation procedure, identifiable with it to lowest order, although the proof is not straightforward. One can rewrite the resulting $\mathscr{H}_{NF}$ in terms of the two action variables I_1 and I_2 (each associated with the u and v oscillators, respectively, of the parabolic coordinate) up to sextet order in z variables as follows:

$$\mathscr{H}_{NF} = \omega(I_1 + I_2)\left\{1 + \frac{\gamma^2}{\omega^4} I_1 I_2 (5 - 4\sin^2(\phi_2 - \phi_1))\right\} \tag{27}$$

where ω is given in (12*b*). The identity of this expression with that of Robnik and Schrüfer is then easy to see. It is a prototype normal-form Hamiltonian for two oscillators in 1:1 resonance, if written in the form

$$\mathscr{H}_{NF} = \omega(I_1 + I_2)\{1 + \lambda^2 I_1 I_2 (1 - k'^2 \sin^2(\phi_2 - \phi_1))\} \tag{28a}$$

where

$$\lambda = \text{real constant} \qquad \text{and} \qquad 0 \leqslant k'^2 \leqslant 1 \ (k'^2 = \tfrac{4}{5} \text{ in (27)}). \tag{28b}$$

This yields a systematic method of 'torus quantisation', which we now discuss.

First, we can replace the two sets of the action-angle variables (I_1, ϕ_1) and (I_2, ϕ_2) by $(\frac{1}{2}(I_1 + I_2), \phi_1 + \phi_2)$ and $(\frac{1}{2}(I_2 - I_1), \phi_2 - \phi_1)$ by noting $\{I_1 \pm I_2, \phi_1 \mp \phi_1\} = 0$, since $\{I_i, \phi_j\} = \delta_{ij}$, so that

$$\mathscr{H}_{NF} = 2n\omega\{1 + \lambda^2(n^2 - I^2)(1 - k'^2 \sin^2\phi)\}. \tag{29}$$

Here

$$I \equiv \tfrac{1}{2}(I_2 - I_1) \qquad \text{and} \qquad \phi \equiv \phi_2 - \phi_1 \tag{30}$$

are a canonical set of the action-angle variables, and the other action

$$n \equiv \tfrac{1}{2}(I_1 + I_2) \tag{31}$$

is a constant of the motion owing to the cyclicity of its conjugate angle variable. Thus, $\mathscr{H}_{NF}$ in (29) contains two constants, namely n (the unperturbed $\mathscr{H}$) and the expression for $\mathscr{H}^{(6)}_{NF}$, i.e.

$$(n^2 - I^2)(1 - k'^2 \sin^2\phi)$$

which should be equated to a newly defined approximate constant. Let us define this by setting the above quantity equal to $n^2k^2(1+\Lambda)(\geqslant 0)$, where Λ is a real parameter expressing a Solov'ev's constant value of $\Lambda = \Lambda(\boldsymbol{A})$ so that

$$n^2k^2(1+\Lambda) = (n^2 - I^2)(1 - k'^2 \sin\phi) \tag{32a}$$

with

$$k^2 \equiv 1 - k'^2 \qquad 0 \leqslant k^2 \leqslant 1 \qquad \text{(see (28}b\text{))}. \tag{32b}$$

Then,

$$I = I(\phi) = n\left(1 - \frac{k^2(1+\Lambda)}{1 - k'^2\sin^2\phi}\right)^{1/2}. \tag{33}$$

The action integral is accordingly given by

$$J = J(\Lambda) = \frac{1}{2\pi}\oint I(\phi)\,d\phi = \frac{2n}{\pi}\int_0^{\phi_0}\sqrt{1 - \frac{k^2(1+\Lambda)}{1-k'^2\sin^2\phi}}\,d\phi \tag{34a}$$

where

$$\phi_0 = \begin{cases} \pi/2 & \text{for } -1 \leqslant \Lambda \leqslant 0 \\ \sin^{-1}\sqrt{1-(k^2/k'^2)\Lambda} & \text{for } 0 < \Lambda \leqslant \Lambda_{max} = k'^2/k^2. \end{cases} \tag{34b}$$

It is now straightforward to quantise the parameter Λ by means of the ordinary quantisation rule of Bohr, Sommerfeld and Maslov. However, it is important to note that the pertinent action integral formula (34a) is not a unique choice, but there exist other expressions of similar formulae. Here we show two such expressions and discuss their mutual relationship:

$$J^{(1)}(\Lambda)=\frac{2n}{\pi}\int_0^{\theta_0}\sqrt{1-\frac{k'^2-k^2\Lambda}{1-k^2\sin^2\theta}}\,\mathrm{d}\theta \tag{35}$$

$$J^{(2)}(\Lambda)=\frac{2n}{\pi}\int_0^{\vartheta_1} \quad\text{and}\quad \frac{n}{\pi}\int_{\vartheta_2}^{\pi-\vartheta_2}\sqrt{1+\frac{\Lambda}{1-k^{-2}\sin^2\vartheta}}\,\mathrm{d}\vartheta. \tag{36}$$

In (35) the upper bound θ_0 of the integration is given by

$$\theta_0=\begin{cases}\sin^{-1}\sqrt{1+\Lambda} & \text{for } -1\leqslant\Lambda\leqslant 0\\ \pi/2 & \text{for } 0<\Lambda\leqslant\Lambda_{\max}=k'^2/k^2.\end{cases} \tag{35a}$$

On the other hand, the range of the integration in $J^{(2)}(\Lambda)$ must be divided into three, i.e. $\vartheta\in[0,\vartheta_1]$, $[\pi-\vartheta_1,\pi]$, and $[\vartheta_2,\pi-\vartheta_2]$ as can be seen in figure 1, which is due to the fact that the integrand becomes infinite at $\vartheta=\vartheta_\infty=\sin^{-1}k<\frac{1}{2}\pi$ and $\pi-\vartheta_\infty$. We shall denote the first and the third parts of $J^{(2)}(\Lambda)$ by $J^{(2)}_{\mathrm{I}}(\Lambda)$ and $J^{(2)}_{\mathrm{III}}(\Lambda)$ so that

$$J^{(2)}_{\mathrm{I}}(\Lambda)=\begin{cases}\dfrac{2n}{\pi}\displaystyle\int_0^{\vartheta_1}\sqrt{1+\frac{\Lambda}{1-k^{-2}\sin^2\vartheta}}\,\mathrm{d}\vartheta & \text{for } -1<\Lambda\leqslant 0\\ 0 & \text{otherwise}\end{cases} \tag{36a}$$

$$J^{(2)}_{\mathrm{III}}(\Lambda)=\begin{cases}\dfrac{2n}{\pi}\displaystyle\int_{\vartheta_2}^{\pi/2}\sqrt{1+\frac{\Lambda}{1-k^{-2}\sin^2\vartheta}}\,\mathrm{d}\vartheta & \text{for } 0<\Lambda\leqslant k'^2/k^2\\ 0 & \text{otherwise.}\end{cases} \tag{36b}$$

These two action integrals are just what Solov'ev obtained from a geometrical consideration to identify the integrand with the component of the angular momentum perpendicular to z, $L_\perp(\vartheta)$, expressed as a function of the polar angle ϑ of the Runge-Lenz vector $\boldsymbol{A}$.

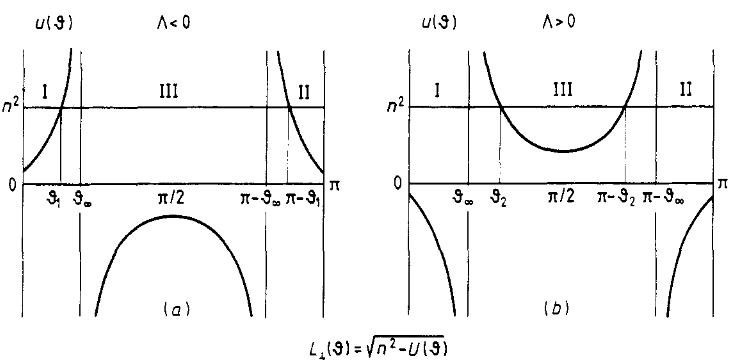

Figure 1. Indication of the ranges of Solov'ev's action integration (36a, b) in terms of the effective potential $U(\vartheta)$: $J^{(2)}(\Lambda)=(1/2\pi)\oint L_\perp(\vartheta)\,\mathrm{d}\vartheta$, $L_\perp(\vartheta)=\sqrt{n^2-U(\vartheta)}$. The three ranges of the integration denoted by I, II and III follow the notation adopted by Cacciani *et al* (1988).

First, we note that the starting action variable I which enters the action integral (34a) is identified with the z component nA_z (see (22')), and that for $L_z = 0$ there are three components of such vectors which could be the possible candidate for the action variable: these are nA_z, $nA_\perp$ and $L_\perp$ ($A_\perp = \sqrt{A_x^2 + A_y^2}$ and $L_\perp = \sqrt{L_x^2 + L_y^2}$). Let us denote them by I, P_1 and P_2, respectively, thus

$$I = nA_z \qquad P_1 = nA_\perp \qquad P_2 = L_\perp . \tag{37}$$

These are not independent of each other, but linked by the fundamental relation (21) of the vectors $\boldsymbol{A}$ and $\boldsymbol{L}$ so that

$$I^2 + P_1^2 + P_2^2 = n^2 \tag{38}$$

($=4/\omega^2$; see (12a), (12b)). It suggests that, besides the action-angle variables (I, ϕ), two other choices of such variables exist in which P_1 and P_2 play the role of relevant action variables, and we show that the two action integral formulae (35) and (36) just provide these choices. We prove this result by constructing the canonical transformations

$$(I, \phi) \rightarrow (P_1, \theta) \qquad \text{and} \qquad (I, \phi) \rightarrow (P_2, \vartheta) \tag{39}$$

explicitly to satisfy the formulae (35) and (36), respectively.

Proof, first step. Let us consider (19a) for $L_z = 0$ which also implies $\phi_1 = \phi_4$ and $\phi_2 = \phi_3$ by recalling (26). This provides the relation

$$P_2^2 (= L_\perp^2) = (n^2 - I^2) \sin^2 \phi \qquad \phi = \phi_2 - \phi_1 \tag{40}$$

also,

$$P_1^2 = n^2 - I^2 - P_2^2 = (n^2 - I^2) \cos^2 \phi. \tag{41}$$

These two relations yield the angle variable ϕ as the function of (I, P_1) and (I, P_2) as follows:

$$\phi = \cos^{-1} \frac{P_1}{(n^2 - I^2)^{1/2}} = \sin^{-1} \frac{P_2}{(n^2 - I^2)^{1/2}}. \tag{42}$$

Second step. Consider a possible generating function $F(I, P_1)$ to derive the canonical transformation $(I, \phi) \rightarrow (P_1, \theta)$, where the old and new coordinates which conjugate to I and P_1 are given, respectively, by

$$\phi = -\frac{\partial F}{\partial I} \qquad \theta = \frac{\partial F}{\partial P_1}. \tag{43}$$

In view of the first equation of (42), such F must be of the form

$$F(I, P_1) = -\int^I \mathrm{d}I \cos^{-1} \frac{P_1}{(n^2 - I^2)^{1/2}} \tag{44}$$

whence

$$\theta = \int^I \mathrm{d}I \frac{1}{(n^2 - I^2 - P_1^2)^{1/2}} = \cos^{-1} \frac{I}{(n^2 - P_1^2)^{1/2}} + \text{constant}. \tag{45}$$

Or, with a proper choice of constant, $I^2=(n^2-P_1^2)\cos^2\theta$. A similar argument shows that for the canonical transformation $(I,\phi)\to(P_2,\vartheta)$ the generating function $F(I,P_2)$ and the new angle variable ϑ are given by

$$F(I,P_2)=-\int^I \mathrm{d}I\,\sin^{-1}\frac{P_2}{(n^2-I^2)^{1/2}} \tag{46}$$

$$\vartheta=\cos^{-1}\frac{I}{(n^2-P_2^2)^{1/2}} \qquad \text{or} \qquad I^2=(n^2-P_2^2)\cos^2\vartheta. \tag{47}$$

Third step. We combine the derived relations (45) (or (47)) with the approximate constant $\Lambda(\boldsymbol{A})$ of Solov'ev to obtain the representation of P_1 (or P_2) as a function of the conjugate angle variable θ (or ϑ). Namely,

$$n^2k^2\Lambda=k'^2P_1^2-k^2I^2 \tag{48}$$

or

$$n^2k^2\Lambda=k'^2(n^2-P_2^2)-I^2 \tag{49}$$

from which we obtain

$$P_1^2=n^2\left(1-\frac{k'^2-k^2\Lambda}{1-k^2\sin^2\theta}\right) \tag{50}$$

$$P_2^2=n^2\left(1+\frac{\Lambda}{1-k^{-2}\sin^2\vartheta}\right). \tag{51}$$

The two action integrals $J^{(1)}=\oint P_1(\theta)\,\mathrm{d}\theta/2\pi$ and $J^{(2)}=\oint P_2(\vartheta)\,\mathrm{d}\vartheta/2\pi$ yield the explicit results (35) and (36), respectively. This concludes our proof.

The most significant conclusion which one can draw from the above proof is the canonical equivalence of the three action integral formulae (34*a*), (35) and (36), and can be expressed simply as

$$I(\phi)\,\mathrm{d}\phi=P_1(\theta)\,\mathrm{d}\theta=P_2(\vartheta)\,\mathrm{d}\vartheta. \tag{52a}$$

Hence the three (actually four) integrals $J(\Lambda)$, $J^{(1)}(\Lambda)$ and $J^{(2)}(\Lambda)(J_{\mathrm{I}}^{(2)}(\Lambda), J_{\mathrm{III}}^{(2)}(\Lambda))$ are equal each to each apart from sign and additional constant factor. This is demonstrated directly in figures 2(*a*–*c*). More precisely, one has

$$J^{(1)}(\Lambda)=n-J(\Lambda) \qquad -1\leqslant\Lambda\leqslant\Lambda_{\max}=k'^2/k^2 \tag{52b}$$

$$J_{\mathrm{I}}^{(2)}(\Lambda)=J^{(1)}(\Lambda) \qquad J_{\mathrm{III}}^{(2)}(\Lambda)=0 \qquad -1\leqslant\Lambda\leqslant 0 \tag{52c}$$

$$J_{\mathrm{III}}^{(2)}(\Lambda)=J(\Lambda) \qquad J_{\mathrm{I}}^{(2)}(\Lambda)=0 \qquad 0\leqslant\Lambda\leqslant\Lambda_{\max}. \tag{52d}$$

The best way to convince ourselves of these results would be to exploit the theory of Jacobian elliptic functions and their integrals, which we have outlined in the appendix. The fact that any of the above action integrals reduces to an elliptic integral (with a common integrand but over different ranges) can be seen by the change of variables: $\sin^2\chi=w$ (χ stands for ϕ, θ and ϑ) in the integrations.

An important background idea for our discussion is that elliptic integrals can be classified in terms of Cauchy integrals of complex, one-valued analytic functions on a doubly connected Riemann surface with genus unity. This topological structure of a dynamics with two degrees of freedom is often cited for two uncoupled harmonic

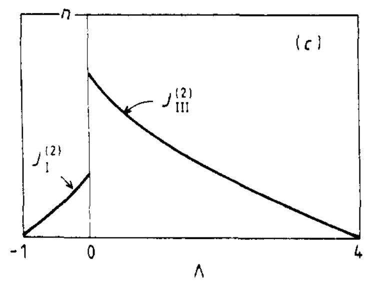

Figure 2. Relations between three action integrals $J(\Lambda)$, $J^{(1)}(\Lambda)$ and $J^{(2)}(\Lambda)$. $J(\Lambda)$ and $J^{(1)}(\Lambda)$ are continuous functions in terms of Λ. The Solov'ev's action integral $J^{(2)}(\Lambda)$ shows discontinuity at $\Lambda = 0$: $J^{(2)}(\Lambda) = J^{(1)}(\Lambda)$ for $\Lambda < 0$, $J^{(2)} = J(\Lambda) = n - J^{(1)}(\Lambda)$ for $\Lambda > 0$.

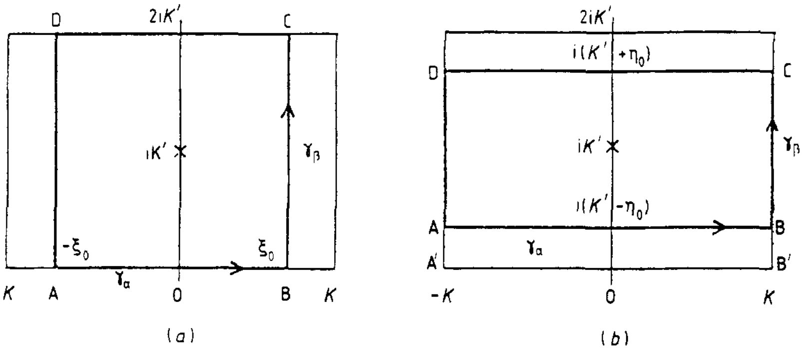

Figure 3. The Riemann surface of the analytic function $P(\zeta)$ of the complex coordinate $\zeta = \alpha + \mathrm{i}\beta + \text{constant}$ on which the two-dimensional two-torus structure is represented; $m = 0$ and (a) $\Lambda < 0$, (b) $\Lambda > 0$. A more detailed description is given in the appendix.

oscillators by the name of KAM torus (e.g. KAM surface in Lichtenberg and Lieberman (1983)). Our result here suggests that, when coupled by a polynomial potential up to fourth order and if the unperturbed oscillators are in 1:1 resonance, the perturbed dynamics within the lowest non-trivial normal form obeys generally the same topology with the action integral written as an elliptic integral (see figure 3).

4. Result and discussion

We present our torus quantisation formula of the diamagnetic Kepler motion which discretises Solov'ev's constant Λ for m (magnetic quantum number) $= 0$:

$$J = J(\Lambda_j) = j + \tfrac{1}{2} \qquad j = 0, 1, \ldots, N-1 \tag{53a}$$

with a given quantised value of $n = N$, $N = 1, 2, \ldots$ (53b)

(equivalent to the Rydberg condition $E=-1/2N^2$). Or, in terms of the other action integral $J^{(1)}$,

$$J^{(1)}=J^{(1)}(\Lambda_{j'})=j'+\tfrac{1}{2} \qquad j'=0,1,\ldots,N-1 \tag{54a}$$

which satisfies $j'=N-1-j$ by virtue of (52b). (54b)

The quantised energy exact up to γ^2 is, accordingly,

$$E_{N,j,m=0}=-\frac{1}{2N^2}+\frac{\gamma^2}{16}N^4(1+\Lambda_j).$$

Specific points of this quantisation condition should be pointed out as follows.

(i) The action integral is bounded for a given constant value $n=N$ (principal quantum number) so that the quantum number j may take only a finite number of allowed values, that is, precisely N values.

(ii) All the α values (the Maslov index) in $J=j+\frac{1}{4}\alpha$ are equal to 2: thus the quantum number which defines a discrete energy spectrum (the quadratic Zeeman spectrum without n mixing) must be half-odd integral.

Our first discussion is to clarify these two points.

The clue to establishing both (i) and (ii) of the above result is given by the identity

$$J(\Lambda)+J^{(1)}(\Lambda)=n \qquad (=N\text{: positive integer}). \tag{55}$$

A proof of this, based on analytic function theory, is presented in the appendix where the two actions $J^{(1)}(\Lambda)$ and $J(\Lambda)$ are shown to be expressed as two Cauchy integrals of a common analytic function but along two distinct elementary cycles characteristic of genus unity, as in (A13) and (A14). Once (55) is admitted, then the boundedness of these actions becomes obvious in view of their positivity: $0\leqslant J(\Lambda), J^{(1)}(\Lambda)\leqslant N$. Further, the quantised values of these actions, i.e. the quantum numbers j for $J(\Lambda)$ and j' for $J^{(1)}(\Lambda)$, must take precisely the same set of values in order that the quantised Λ thus determined in two-fold ways be identical. The identity (55) then implies a reflection symmetry of the finite set of the quantised action in the interval $[0, n]$ about its midpoint.

Consequently, two possibilities arise about the Maslov index α for j and α' for j': either $\alpha=\alpha'=0$ or $\alpha=\alpha'=2$, and $j+j'=N$ in the former case and $j+j'=N-1$ in the latter. But, the former case is excluded because the action integration in the representations (A13), (A14) involves real caustic points (the four vertex points A, B, C, D, in figure 3; see also a subtlety about these caustic points discussed after (A13) and (A14) in the appendix). This is our special emphasis, because several authors who have treated the problem have arrived at the same conclusion (for example, Cacciani *et al* (1988) and Waterland *et al* (1987)) but without giving a convincing reason. A confusing point in this problem is that the pertinent action integration involves two separate regions; one associated with the 'libration' character and the other with the 'rotation' character; for example, in the integration (35) for $J^{(1)}(\Lambda)$ the region $\Lambda\in[-1,0]$ is associated with *libration* and $\Lambda\in[0,\Lambda_{max}]$ with *rotation* (see an article by Delande and Gay in Taylor (1988)). But this should not admit the distinction between the Maslov index value for quantising Λ with a full-integral and with a half-odd quantum number, thus making the concept of *libration against rotation* meaningless in the present example. The convincing reason we find for this result is that there exist two action integrals, quantising Λ in an 'ascending' way and a 'descending' way, both equally allowed to give the same quantised values.

Solov'ev (1982) has proposed his torus quantisation formula in the same problem in terms of the canonical set (ϑ, $L_\perp$), which reads (for $p_\phi = 0$)

$$J_{\mathrm{I}}^{(2)}(\Lambda_k) = k + \tfrac{1}{2} \qquad k = 0, 1, 2, \ldots \tag{56a}$$

$$J_{\mathrm{III}}^{(2)}(\Lambda_{k'}) = k' + \tfrac{1}{2} \qquad k' = 0, 1, 2, \ldots. \tag{56b}$$

The reason for this puzzling feature of a 'two-fold' quantum number can be visualised in figure 2(c): that is to say, the action integral $J^{(2)}(\Lambda)$ is a discontinuous function of Λ at $\Lambda = 0$ (52c, d). An obvious remedy for this discontinuity is to replace either one or the other portion of the function, $J_{\mathrm{I}}^{(2)}(\Lambda)$ (or $J_{\mathrm{III}}^{(2)}(\Lambda)$), by $n - J_{\mathrm{I}}^{(2)}(\Lambda)$ (or $n - J_{\mathrm{III}}^{(2)}(\Lambda)$), making it identical to $J(\Lambda)$ (or $J^{(1)}(\Lambda)$). This has been done, indeed, in the analysis of experiments carried out by Cacciani *et al* (1988) on an intuitive basis.

Finally, we add a brief discussion about higher-order perturbation effects on the KAM tori. This is not a systematic treatment of the BG normal forms: Instead, an obvious consideration that an analytic simulation of the tori could be made by taking

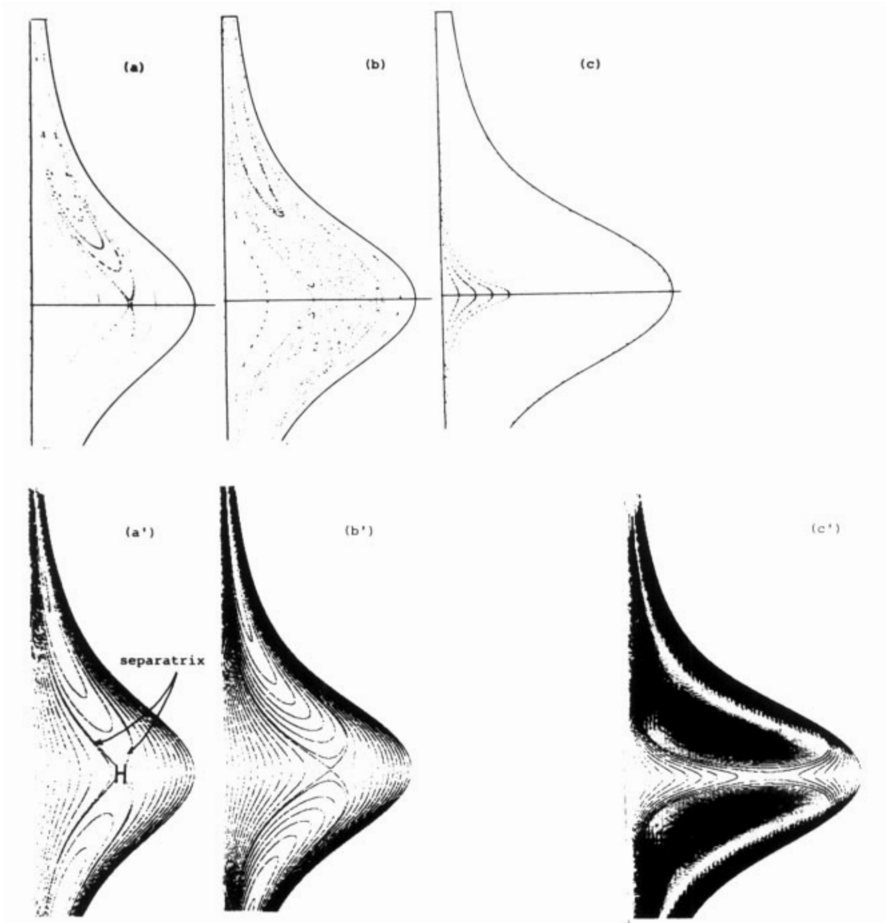

Figure 4. The Poincaré surface of section of $z = 0$, $P_\rho - \rho$ plane in the cylindrical coordinate system for the diamagnetic Kepler motion: (a), (b), (c) are those computed by integrating the equations of motion, and (a)′, (b)′, (c)′ are those simulated by taking intersections of $\mathscr{H}_{\mathrm{NF}}$ and $\Lambda(\boldsymbol{A}) = \Lambda$. ($a$)-($a$)′ $m = 0$, $B = 2$, $E = -1$; (b)-(b)′ $m = 0$, $B = 2$, $E = -0.7$; (c)-(c)′ $m = 0$, $B = 2$, $E = -0.3$ in the atomic unit.

intersections between the two constants is extended within the lowest non-trivial normal form but one step higher than that treated so far (i.e. the intersections between $\mathscr{H}^{(2)}_{\rm NF}=4$ and $\Lambda(\boldsymbol{A})=\Lambda$). Thus we simulate them by taking the intersections between $\mathscr{H}_{\rm NF}=4$ in (23*a*) and $\Lambda(\boldsymbol{A})=\Lambda$ in (23*b*). Figures 4(*a*–*c*) show such drawings, which are compared with the surface of sections computed from the diamagnetic Kepler trajectories: these demonstrate the coexistence of chaos and undestroyed tori (so-called 'remnants', Reinhardt and Farrelly (1982)) from our previous studies (Hasegawa *et al* (1984)).

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Appendix. Jacobian elliptic function and the proof of equalities in (52*a*–*d*) based on Erdélyi *et al* (1953)

Our starting point is the definition of the *sn* function

$$w=sn(u,k) \tag{A1}$$

by

$$u=\int_0^w[(1-x^2)(1-k^2x^2)]^{-1/2}\,\mathrm{d}x=\int_0^{\sin^{-1}w}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{A2a}$$

as the solution to the first-order differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}u}=[(1-w^2)(1-k^2w^2)]^{1/2}\qquad w(0)=0. \tag{A2b}$$

The other related functions are $cn(u,k)$ and $dn(u,k)$ which satisfy

$$cn^2(u,k)+sn^2(u,k)=1\qquad dn^2(u,k)+k^2sn^2(u,k)=1. \tag{A3}$$

The parameter k is said to be the *modulus* and $k'=\sqrt{1-k^2}$ the *complementary modulus*. The first-class elementary properties of these functions are:

$$\frac{\mathrm{d}}{\mathrm{d}u}sn(u,k)=cn(u,k)dn(u,k) \tag{A4}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}cn(u,k)=-sn(u,k)dn(u,k) \tag{A5}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}dn(u,k)=-k^2sn(u,k)cn(u,k) \tag{A6}$$

and

$$sn(-u,k)=-sn(u,k)\qquad cn(-u)=cn(u)\qquad dn(-u)=dn(u). \tag{A7}$$

Another class of the basic properties is that a change of the modulus

$$k\to k'\quad\text{or}\quad k\to k^{-1}\quad\text{or}\quad k\to k'^{-1}\quad\text{or}\quad k\to \mathrm{i}k/k'\quad\text{or}\quad k\to k'/\mathrm{i}k$$

in an elliptic integral gives rise to a linear combination of elliptic integrals with the original modulus k. The simplest example is given by

$$sn(u, k^{-1}) = ksn(u/k, k). \tag{A8}$$

We can now show a direct proof of the canonical equivalence of the action angle variables (I, ϕ), (P_1, θ) and (P_2, ϑ) expressed in (52a), i.e.

$$\left(1 - \frac{k^2(1+\Lambda)}{1-k'^2\sin^2\phi}\right)^{1/2} d\phi = \left(1 - \frac{k^2 - k'^2\Lambda}{1-k^2\sin^2\theta}\right)^{1/2} d\theta \tag{A9a}$$

$$= \left(1 + \frac{\Lambda}{1-k^{-2}\sin^2\vartheta}\right)^{1/2} d\vartheta. \tag{A9b}$$

Proof. Let $\sin\phi = dn(u, k)/k'$ in the left-hand side of (A9a). Then, $\cos\phi\, d\phi = -(k^2/k')sn(u, k)cn(u, k)\, du$ by differentiating it. But, since

$$\cos\phi = (1-\sin^2\phi)^{1/2} = \left(1 - \frac{1}{k'^2} dn^2(u, k)\right)^{1/2}$$

$$= \frac{1}{k'}(k'^2 - dn^2(u, k))^{1/2} = \frac{ik}{k'} cn(u, k)$$

one gets

$$d\phi = iksn(u, k)\, du = i(1 - dn^2(u, k))^{1/2}\, du$$

$$= i(1 - k'^2\sin^2\phi)^{1/2}\, du \tag{A10}$$

hence

$$\frac{d\phi}{[1-k'^2\sin^2\phi]^{1/2}} = i du.$$

Thus,

$$\left(1 - \frac{k^2(1+\Lambda)}{1-k'^2\sin^2\phi}\right)^{1/2} d\phi = i[k'^2 - k^2\Lambda - dn^2(u, k)]^{1/2}\, du$$

$$= [k^2(1+\Lambda) - k^2 sn^2(u, k)]^{1/2}\, du \tag{A11}$$

and by setting $sn(u, k) = \sin\theta$ with $du = d\theta/[(1-k^2\sin^2\theta)^{1/2}]$ the first relation (A9a) can be deduced. In order to get the second relation (A9b), we utilise (A8) in (A11) just obtained:

$$\left(1 - \frac{k^2 - k'^2\Lambda}{1-k^2\sin^2\theta}\right)^{1/2} d\theta = [k^2(1+\Lambda) - k^2 sn^2(u, k)]^{1/2}\, du$$

$$= [\Lambda + 1 - sn^2(k^{-1}u', k)]^{1/2}\, du'$$

$$= [\Lambda + 1 - k^{-2} sn^2(u', k^{-1})]^{1/2}\, du'$$

$$= \left(1 + \frac{\Lambda}{1-k^{-2}\sin^2\vartheta}\right)^{1/2} d\vartheta$$

by setting $sn(u', k^{-1}) \equiv \sin\vartheta$, thus establishing (A9$b$).

We now make our discussion more transparent from a viewpoint of analytic function theory of a complex variable (see Lakshmanan and Hasegawa 1984, Hasegawa and

Adachi 1988). The argument is that the diamagnetic Kepler motion can be separated, in the weak-field limit, in terms of the elliptic cylindrical coordinates (Hasegawa *et al* 1989), where the mapping of the two coordinates (α, β) to their momenta (P_α, P_β) can be singled out as a complex analytic function, $\zeta \to P_\zeta = P(\zeta)$:

$$\zeta = \alpha + \mathrm{i}\beta \qquad P_\zeta = P_\alpha + \mathrm{i}P_\beta$$

and for $P_\phi(=p_\phi) = 0$

$$P_\zeta = [n^2k^2(1+\Lambda) - n^2k^2 sn^2(\zeta, k)]^{1/2} \equiv P(\zeta). \qquad \text{(A12)}$$

This function is two-valued on the complex ζ plane and regular except at $\zeta = \mathrm{i}K'$ ($K' = K(k')$: the complete elliptic integral of the first kind with modulus k' inside of the fundamental parallelogram of $sn(\zeta, k)$). The Reimann surface on which $P(\zeta)$ is representable as single-valued can be synthesised by joining two copies of the ζ plane at two pairs of the cuts connecting four zeros of $P(\zeta)$ (A, B, C, D in figure 3(*a*) or (*b*). The two elementary cycles (closed path of Cauchy integration) to be chosen on this torus with genus 1 for the action integration $\int P(\zeta)\,\mathrm{d}\zeta$ are denoted by γ_α and γ_β, which can be identified with the segments AB and BC, respectively, in the figure. One has, indeed,

$$J^{(1)}(\Lambda) = \frac{1}{2\pi}\int_{\gamma_\alpha} P(\zeta)\,\mathrm{d}\zeta = \frac{2n}{\pi}\int_{\mathrm{AB}} [k^2(1+\Lambda) - k^2 sn^2(\alpha, k)]^{1/2}\,\mathrm{d}\alpha \qquad \text{(A13)}$$

$$J(\Lambda) = \frac{1}{2\pi}\int_{\gamma_\beta} P(\zeta)\,\mathrm{d}\zeta = \frac{2n}{\pi}\int_{\mathrm{BC}} [k^2(1+\Lambda) - dn^2(\beta, k')]^{1/2}\,\mathrm{d}\beta. \qquad \text{(A14)}$$

More precisely, in (A13)

$$\int_{\mathrm{AB}} \cdot\,\mathrm{d}\alpha \equiv \int_{-\xi_0}^{\xi_0} \cdot\,\mathrm{d}\alpha \qquad \text{for } \Lambda < 0$$

and

$$\int_{\mathrm{AB}} \cdot\,\mathrm{d}\alpha = \int_{-K}^{K} \cdot\,\mathrm{d}\alpha \qquad \text{for } \Lambda > 0$$

and in (A14)

$$\int_{\mathrm{BC}} \cdot\,\mathrm{d}\beta \equiv \int_{0}^{2K'} \cdot\,\mathrm{d}\beta \qquad \text{for } \Lambda > 0$$

and

$$\int_{\mathrm{BC}} \cdot\,\mathrm{d}\beta = \int_{K'-\eta_0}^{K'+\eta_0} \cdot\,\mathrm{d}\beta \qquad \text{for } \Lambda > 0$$

in which $\pm\xi_0$ denote the two possible zeros of the integrand on the real axis inside $[-K, K]$ and $K' \pm \eta_0$ for those on the shifted imaginary axis, $\zeta = K + \mathrm{i}\beta$, inside $[0, 2K']$ as to the β-variable. To assure these representations it is enough to exploit the following two facts pertaining to the Jacobian elliptic functions.

(i) Identity

$$sn(K + \mathrm{i}K' + \mathrm{i}\beta, k) = \frac{1}{k}\, dn(\mathrm{i}\beta, k)/cn(\mathrm{i}\beta, k)$$
$$= \frac{1}{k}\, dn(\beta, k') \qquad \text{(A15)}$$

(see Erdelyi *et al* (1953) pp 344 and 346).

(ii) The analytic function $P(\zeta)$ defined in (A12), outside of the rectangle ABCD (on each sheet of the Riemann surface) but inside of the fundamental parallelogram, is one-valued and regular.

This latter fact enables one to excute a parallel shift of the integration: for example for $\Lambda > 0$, from that on the segment A'B' to the other on AB, i.e.

$$\frac{2}{\pi}\int_{\mathrm{A'B'}} P(\alpha)\,\mathrm{d}\alpha = \begin{cases} J^{(1)}(\Lambda) & \text{in (35}a\text{) (35}b\text{)} \qquad \theta_0 = \dfrac{\pi}{2} \\ \dfrac{2}{\pi}\displaystyle\int_{\mathrm{AB}} P(\alpha)\,\mathrm{d}\alpha & \text{in (A13).} \end{cases} \tag{A16}$$

For this, it is remarked that the contour integral of $P(\zeta)$ along the closed cycle $\overrightarrow{\mathrm{A'B'BAA'}}$ vanishes, and that

$$\int_{\overrightarrow{\mathrm{B'B}}} P(\zeta)\,\mathrm{d}\zeta + \int_{\overrightarrow{\mathrm{AA'}}} P(\zeta)\,\mathrm{d}\zeta = 0 \tag{A17}$$

holds due to the periodicity of the *sn* function (and hence of the function $P(\zeta)$ in one sheet) with period $2K$. Observe, also, that by this parallel shift the representation (A13) of the action integral becomes such that it involves two caustics (i.e. the square-root zeros A and B of the integrand).

The same kind of the parallel shift holds also for the β integration, for which the identity (A16) is used to relate $J(\Lambda)$ in (34) with (A14).

Upon establishing the representations (A13) and (A14), it is now easy to prove the identity

$$J^{(1)}(\Lambda) + J(\Lambda) = n \tag{A18}$$

because, then, $J^{(1)} + J(\Lambda)$ is identical to the contour integral of the single-valued analytic function $P(\zeta)$ on the Riemann surface along ABCDA with residue n which arises from the unique simple pole $\zeta = \mathrm{i}K'$ inside the contour.

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